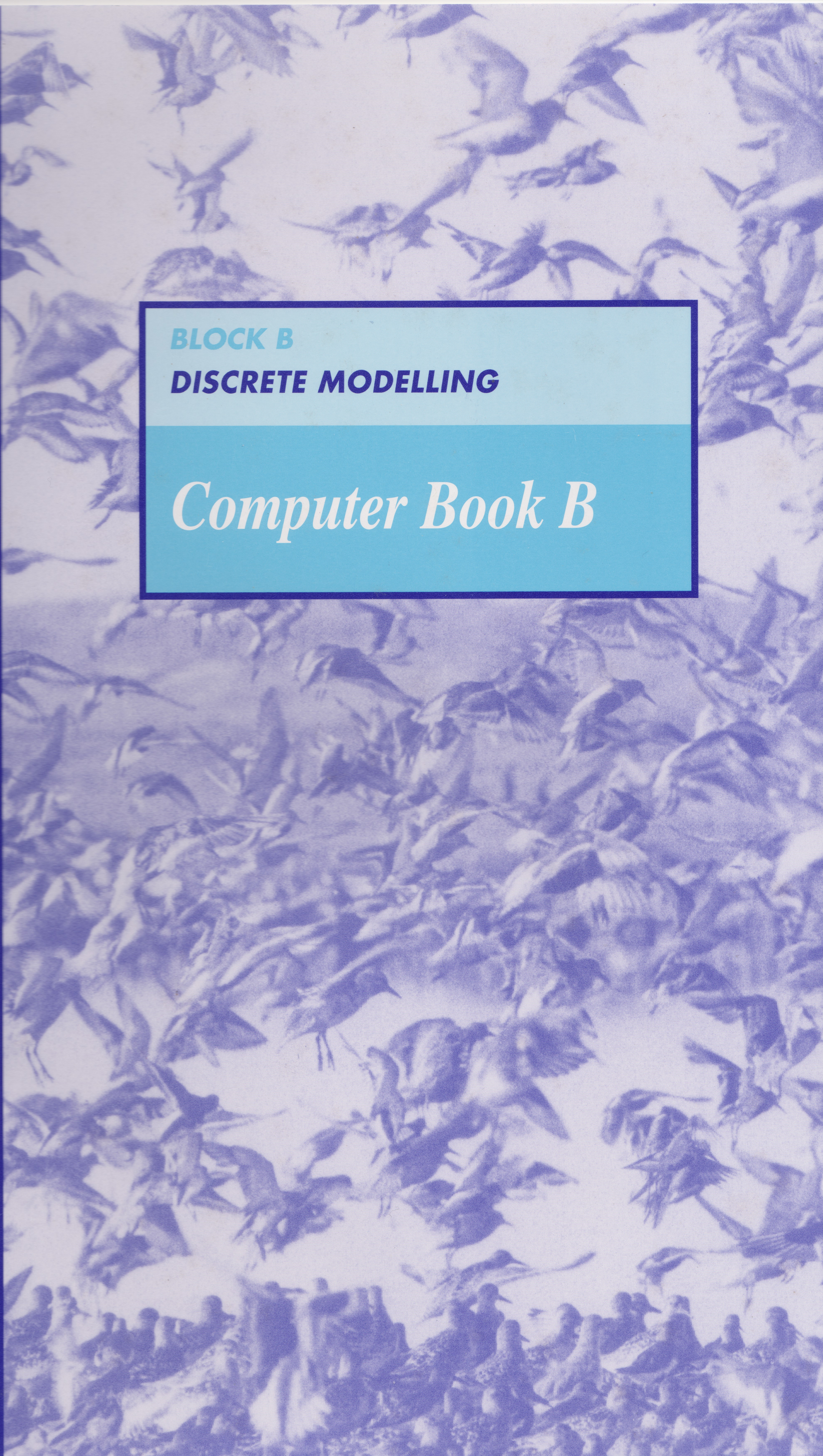
MST121 CB B

The Open University

A first level interdisciplinary course

Using **Mathematics**



BLOCK B

DISCRETE MODELLING

Computer Book B

COMPUTER BOOK

B

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Using Mathematics

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Prepared by the course team

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About this course

This computer book forms part of the course MST121 *Using Mathematics*. This course and the courses MU120 *Open Mathematics* and MS221 *Exploring Mathematics* provide a flexible means of entry to university-level mathematics. Further details may be obtained from the address below.

MST121 uses the software program Mathcad (MathSoft, Inc.) and other software to investigate mathematical and statistical concepts and as a tool in problem solving. This software is provided as part of the course.

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Contents

Guidance notes

This computer book contains those sections of the chapters in Block B which require you to use Mathcad. Each of these chapters contains instructions as to when you should first refer to particular material in this computer book, so you are advised not to work on the activities here until you have reached the appropriate points in the chapters.

In order to use this computer book, you will need the following Mathcad files.

Chapter B1

121B1-01 Logistic recurrence sequences
121B1-02 Overview of logistic recurrence sequences (Optional)

Chapter B2

121B2-01 Matrices in Mathcad
121B2-02 Exploring two subpopulations
121B2-03 Exploring three subpopulations

Chapter B3

There are no specified computer activities associated with Chapter B3.

Instructions for installing these files onto your computer's hard disk, and for opening them, are given in Chapter A0.

Activities based on software vary both in nature and in length. Sometimes the instructions for an activity appear only in the computer book; in other cases, instructions are given in the computer book and on screen. Feedback on an activity is sometimes provided on screen and sometimes given in the computer book.

For advice on how each computer session fits into suggested study patterns, refer to the Study guides in the chapters.

Chapter B1, Section 4
Logistic recurrence sequences on the computer

In this section, you will use the computer to investigate the behaviour in the long term of logistic recurrence sequences. There are two Mathcad files accompanying this section. The first file plots graphs of logistic recurrence sequences and enables the effect of altering parameters in the recurrence system to be investigated. The second file, which is *optional*, provides an overview of the types of long-term behaviour that can be obtained from a logistic recurrence sequence.

4.1 Investigating logistic recurrence sequences

The logistic model for population variation is described by the recurrence relation

See Chapter B1, Subsection 3.1.

$$P_{n+1} - P_n = rP_n\left(1 - \frac{P_n}{E}\right) \quad (n = 0, 1, 2, \ldots),$$

where P_n is the population size n years after some chosen starting time. The initial population size is P_0, and the other parameters are E, the equilibrium population level, and r, the proportionate growth rate at small population sizes.

In the first activity you will use the computer to investigate the effect of altering the value of P_0 on the long-term behaviour of the recurrence sequence. Then, in Activities 4.2 and 4.3, you will investigate the effect of altering the parameter r.

As you may recall, the parameter E is just a scaling factor, so no new types of long-term behaviour arise from altering the value of E.

See Chapter B1, Subsection 3.3.

Activity 4.1 Altering the initial population size

Open Mathcad file **121B1-01 Logistic recurrence sequences**. Page 1 introduces the worksheet. Look at page 2, where the task starts with the logistic model for the population of barnacle geese at Caerlaverock, Dumfries. The parameters for this model are $P_0 = 3200$, $r = 0.2$ and $E = 13\,300$. A table of values of terms of the sequence is given on this page, along with a fixed-scale graph that illustrates the sequence. Only terms of the sequence with values between 0 and 20 000 (shown as $2 \cdot 10^4$) can be shown on this scale.

Remember to make your own working copy of the file.

This population was introduced in Example 3.1 of Chapter B1.

Recall that a scroll bar will appear when you click on a value in the table.

Investigate the effect on the sequence of changing the value of the initial population size P_0, while keeping the parameters r and E constant. Set $N = 50$, so that longer-term behaviour can be observed. Use in turn the following values for P_0:

$$20\,000,\ 10\,000,\ 5000,\ 1000,\ 100.$$

Describe the long-term behaviour of these sequences, recording your observations in words and with small sketches. For example,

'the sequence is increasing and tends to E'

and Figure 4.2, when taken together, adequately describe the long-term behaviour of the sequence with $P_0 = 3200$.

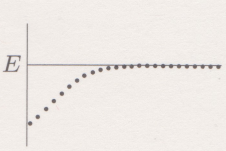


Figure 4.2 Small sketch to describe the long-term behaviour of a sequence

Do these sequences all tend to the same value?

Solutions are given on page 22.

Comment

See Chapter B1, Subsection 3.3.

The long-term behaviour of the sequence is *not* affected by the initial population size P_0, provided that P_0 lies between 0 and $E(1+1/r)$. For any such value of P_0, the sequence tends to E.

For the logistic recurrence relation with $r = 0.2$, we can say the following.

◇ For an initial population size P_0 below E but greater than 0, in this case $0 < P_0 < 13\,300$, the population increases and approaches the equilibrium level, $E = 13\,300$. The population then remains effectively constant at E.

In fact, if P_0 is above E but below $E/r = 66\,500$, then the sequence decreases throughout. If P_0 is between E/r and $E(1+1/r)$, then the second term of the sequence is less than E, and thereafter the sequence behaves as for the case with $0 < P_0 < E$.

◇ For an initial population P_0 above E but below $E(1+1/r)$, in this case $13\,300 < P_0 < 79\,800$, the population decreases initially and approaches the equilibrium level, $E = 13\,300$. Again, the population then remains effectively constant at E.

◇ If the initial population size P_0 is equal to the equilibrium level E, then the population remains constant at this value.

◇ If the initial population size P_0 is equal to $E(1+1/r)$, in this case $P_0 = 79\,800$, then $P_n = 0$ for all $n \geq 1$; that is, all subsequent terms of the sequence are zero. If the initial population size P_0 is greater than $E(1+1/r)$, then the next term of the sequence is negative. Therefore, the model predicts that for initial population sizes of $E(1+1/r)$ or more, the population becomes extinct within a year.

You may like to try in turn the values $P_0 = 79\,800$, $P_0 = 79\,810$ and $P_0 = 79\,790$. The table shows the detailed results in each case. In order to see all non-negative terms of these sequences on the graph, you will need to rescale the vertical axis, by setting $Y2 = 80\,000$ (say). For the case $P_0 = 79\,790$, the sequence shows a sharp drop in the first year, but the population then recovers and tends to $E = 13\,300$.

This is an extreme example of the type of behaviour described above for P_0 between E/r and $E(1+1/r)$.

Notice that the values in the table are given to 3 decimal places. When dealing with a population, as here, we round these values to the nearest whole number. This would not be necessary, however, if we were modelling a very large population and counting in millions.

Remember that Mathcad notes are *optional*.

Mathcad notes

◇ The range variable n used to calculate the terms of the sequence is defined as $n := 0, 1 .. N-1$. This ensures that the subsequent definitions of the subscripted variables P_0 and P_{n+1}, given by

$$P_0 := 3200, \quad P_{n+1} := P_n \left[1 + r\left(1 - \frac{P_n}{E}\right)\right],$$

together define *all* the terms $P_0, P_1, \ldots, P_N$ and *no others*. Mathcad carries out the definition for P_{n+1} here N times, once for each of the values $0, 1, \ldots, N-1$ in the range of n. As it does so, the subscript $n+1$ takes each of the values in the range $1, 2, \ldots, N$ in turn.

◇ Mathcad variables can be defined more than once, so they can take different values at different places in the worksheet. For example, the range variable n is defined as $n := 0, 1 .. N-1$ at the top of page 2 (to calculate the terms of the sequence) and as $n := 0, 1 .. N$ lower down the page (to plot them on the graph).

◇ The vertical scale on the graph is fixed from $Y1$ to $Y2$. This has been done by defining two variables $Y1$ and $Y2$ above the graph, then

entering $Y1$ and $Y2$ into the axis limit placeholders. The graph can thus be rescaled by changing the value of $Y1$ or $Y2$, without the need to edit the graph itself. (The horizontal scale has also been fixed, from 0 to N.)

You are now asked to use the computer to investigate the effect of altering the parameter r. Taking values of r greater than 3 gives sequences whose terms quickly become very large, so we concentrate on values in the interval $(0, 3]$.

Interval notation was introduced in Chapter A3, Subsection 1.1.

Activity 4.2 Altering the parameter r

Investigate the effect of altering the parameter r on the long-term behaviour of the logistic recurrence sequence. Set $P_0 = 5000$, $N = 50$ and leave $E = 13\,300$.

You should still be working with Mathcad file 121B1-01. If you previously altered the value of $Y1$ or $Y2$, then these should now be reset to $Y1 = 0$ and $Y2 = 20\,000$.

Use in turn the following values for r:

0.5, 1, 1.5, 2, 2.5, 3.

In some cases (in particular, for $r = 2$, 2.5 and 3), you may find it helpful to increase the value of N to 100, 500 or perhaps even more.

Record your observations in words and with small sketches.

Solutions are given on page 22.

Comment

Different values of r show markedly different types of behaviour.

- For $r = 0.5$ and $r = 1$, the sequence increases and tends to $E = 13\,300$. This is the same type of behaviour as displayed by the sequence in Activity 4.1, where $P_0 = 5000$, $E = 13\,300$ and $r = 0.2$.
- For $r = 1.5$ and $r = 2$, the terms of the sequence alternate between values above and below $E = 13\,300$, gradually approaching E. Hence the sequence tends to E.

In the case of $r = 2$, the convergence to E is very slow and can be seen convincingly only by choosing a large value for N.

- For $r = 2.5$, the terms of the sequence again alternate between values above and below $E = 13\,300$. In this case, however, the terms *do not* approach E. In the long run, each term is close to one of *four* values: 9326, 16 292, 7128 and 15 398 (to the nearest integer). This type of behaviour, of repeatedly taking a number of values in order, is called **cycling**. Here four repeating values are involved, so we have a **4-cycle**.

For example, these are the rounded values of P_{91}, P_{92}, P_{93}, P_{94}, respectively. They repeat as P_{95}, P_{96}, P_{97}, P_{98}, and so on.

- For $r = 3$, another type of long-term behaviour is displayed. Again, terms of the sequence do not approach $E = 13\,300$, as can be seen clearly by increasing N to 500 or 1000. The sequence behaves in an unpredictable manner, taking values which seem to be random.

  However, we can at least say that terms of the sequence lie in the range $0 < P_n < 20\,000$, although, of course, no term is actually equal to $E = 13\,300$. This type of apparently unstructured behaviour is referred to as **chaotic**.

In fact, it looks as if P_n is always less than 18 000.

The values for P_0 and E were chosen for convenience and clarity. Similar results would be found using other values of E and values of P_0 between 0 and $E(1 + 1/r)$.

The six values of r used in Activity 4.2 were chosen to be equally spaced in the range from 0 to 3. The next step in a systematic investigation of this type is to pick values of r in the intervals between those pairs of values from Activity 4.2 which gave different types of long-term behaviour. For example, the three values 1.25, 2.25 and 2.75 are sensible choices. We ask you to continue this investigation in Activity 4.3.

Activity 4.3 Using further values for the parameter r

You should still be working with Mathcad file 121B1-01.

Investigate further the effect of altering the parameter r on the long-term behaviour of the logistic recurrence sequence. Again set $P_0 = 5000$, $N = 50$ and $E = 13\,300$.

Use in turn the following values for r:

1.25, 2.25, 2.75.

Record your observations in words and with small sketches. (Remember that you may have to increase the value of N to observe the long-term behaviour.)

Solutions are given on page 23.

Comment

Only one new type of behaviour is shown by sequences with these values of r.

- ◇ For $r = 1.25$, the terms of the sequence alternate between values just above and just below $E = 13\,300$, and the sequence tends to E. This is the same type of long-term behaviour as displayed in Activity 4.2 by the sequences with $r = 1.5$ and $r = 2$.
- ◇ For $r = 2.25$, the terms of the sequence alternate between values above and below $E = 13\,300$. The sequence *does not* tend to E, but settles to one value (15 608) above E and one value (9515) below E. We describe this type of long-term behaviour as a **2-cycle**.
- ◇ For $r = 2.75$, the terms of the sequence take values which seem to be random between two bounds. The long-term behaviour is chaotic, as displayed in Activity 4.2 by the sequence with $r = 3$.

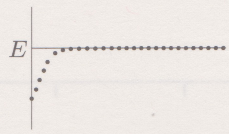
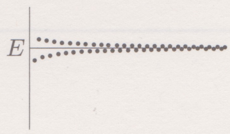
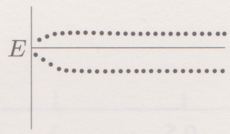
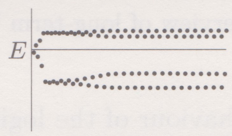
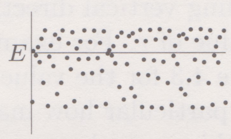
Both values here are rounded to the nearest integer.

Now close file 121B1-01.

On the basis of the cases that you examined using Mathcad in Activities 4.2 and 4.3, the table on page 9 can be drawn up to describe the long-term behaviour of the sequences.

Further investigation would be required to determine a fuller range of behaviour. It would clearly be possible to obtain more information by trying many more values of r, in an attempt to ‘fill in the gaps’. In some cases this can give rapid answers. For example, it does not take much experimentation to establish with some confidence that the behaviour described in the table for $r = 0.5$ and $r = 1$ is shared by all values of r in the interval $(0, 1]$, whereas any value in the interval $(1, 2]$ gives the behaviour described for $r = 1.25$, 1.5 and 2.

However, rather than looking at a succession of individual cases, we really need some sort of ‘overview’ of what is going on. This is the topic of the next subsection.

r	Long-term behaviour of P_n	Sketch graph
0.5, 1	Tends to E, with values always just below E	
1.25, 1.5, 2	Tends to E, with values alternating between just above and below E	
2.25	2-cycle, with one value above E and one value below E	
2.5	4-cycle, with two values above E and two values below E	
2.75, 3	Chaotic variation between bounds	

4.2 An overview of long-term behaviour

Rather than look in turn at the sequences determined by particular values of r, it is possible to obtain an overview of the long-term behaviour of the sequences as the parameter r varies. Computers are most appropriately used for tasks that involve a relatively simple specification but much calculation, and this is very much a case in point.

The recurrence relation

$$x_{n+1} = x_n + rx_n(1 - x_n) \quad (n = 0, 1, 2, \ldots),$$

This is derived in Chapter B1, Subsection 3.3.

in which the parameter E does not appear, is used for simplicity. This is the logistic recurrence relation with $E = 1$ or, alternatively, the logistic recurrence relation rewritten using the substitution $x_n = P_n/E$.

In the previous two activities, the terms of each sequence were plotted from left to right, as a graph of P_n against n, giving a different graph for each value of the parameter r used. We now construct *one* plot that indicates the variety of types of long-term behaviour that can occur.

This overview is obtained as follows. A given starting value x_0 is defined. For each value of r in the specified range, the first 301 terms of the sequence, $x_0, x_1, \ldots, x_{300}$, are calculated, and then the first 200 terms, $x_0, x_1, \ldots, x_{199}$, are discarded. The remaining terms of the sequence are then plotted against the value of the parameter r. So, for each value of r, the graph displays as a small dot each of the points (r, x_{200}), (r, x_{201}), ..., (r, x_{300}).

Figure 4.3 shows this type of overview as generated by Mathcad. The values of the parameter r used for this graph are $0, 0.002, 0.004, \ldots, 3$.

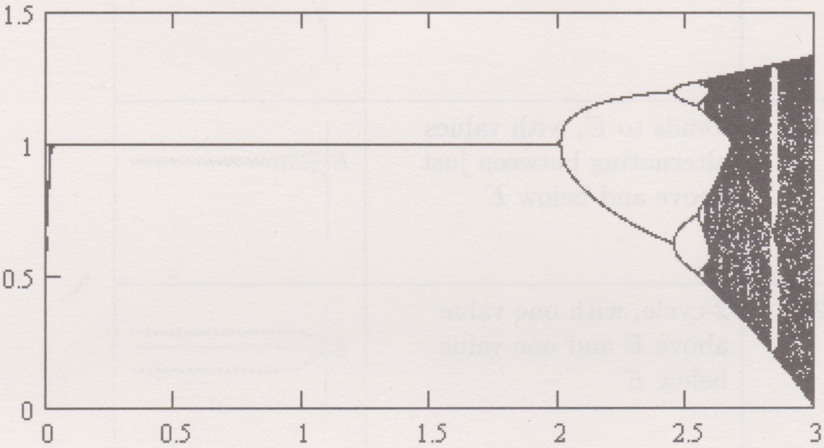


Figure 4.3 Overview of long-term behaviour of logistic recurrence sequences for $0 \le r \le 3$

To find the behaviour of the logistic recurrence sequence for a given value of r, you look along the horizontal axis to find the value of r, then look up the corresponding vertical direction at the values of the sequence plotted against this value of r. For example, look along the horizontal axis of the graph in Figure 4.3 for the value $r = 1.5$. Then look vertically to see which points, and in particular how many points, have been plotted against this value of r. In this case, there appears to be just one point plotted: $(1.5, 1)$. This indicates that (to the accuracy shown on the graph) all of the terms $x_{200}, x_{201}, \ldots, x_{300}$ of this sequence are equal to 1; that is, for this value of r, the sequence tends to the equilibrium population level $E = 1$, as expected from our previous investigations.

It can be seen from Figure 4.3 that the more complicated behaviour occurs for values of r greater than 2. Figure 4.4 shows another Mathcad-generated overview. For this graph, the values for the parameter r are $2.4, 2.4004, 2.4008, \ldots, 3$.

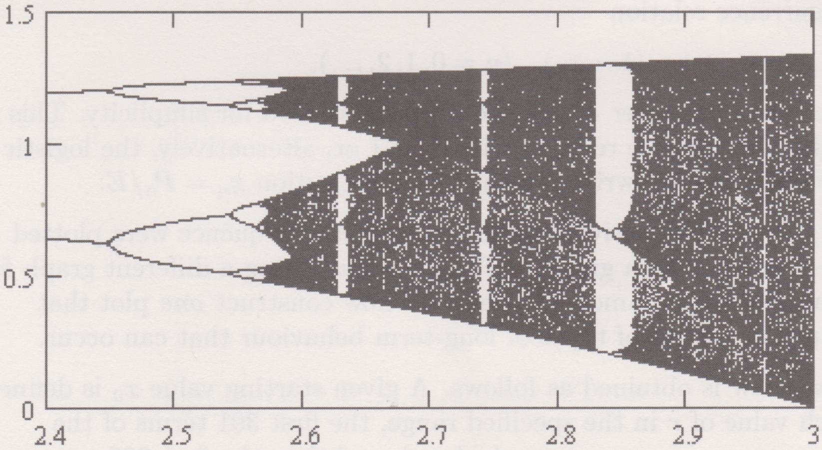


Figure 4.4 Overview of long-term behaviour of logistic recurrence sequences for $2.4 \le r \le 3$

Look along the horizontal axis of the graph in Figure 4.4 for the value $r = 2.5$, then look vertically to see how many points have been plotted against this value. In this case there appear to be four points plotted, two above and two below the value $E = 1$. This means that each of the terms $x_{200}, x_{201}, \ldots, x_{300}$ of this sequence takes one of these four values. Our findings from the previous investigations confirm that the long-term behaviour of the sequence with $r = 2.5$ is a 4-cycle.

In general, we can deduce that if there are a small number of points plotted on the vertical line for a given value of r, then the terms of the corresponding sequence approach a cycle with this number of values. For example, it looks as if there may be values of r between 2.5 and 2.6 where the long-term behaviour of the sequence is an 8-cycle.

For many values of r, a 'smear' of points is plotted in Figure 4.4. This indicates chaotic behaviour, where the terms $x_{200}, x_{201}, \ldots, x_{300}$ of the sequence take seemingly random values within two bounds. One of these bounds is above $E = 1$, and the other is below. Each of the bounds varies (smoothly) as r varies.

Note that the regime for chaos, beginning somewhere between $r = 2.5$ and $r = 2.6$, does *not* continue unbroken to $r = 3$. Indeed, there are values of r between 2.8 and 2.9 where it appears that the long-term behaviour of the sequence may be a 3-cycle or perhaps even a 6-cycle!

The following optional activity investigates further this use of Mathcad to provide an overview of the long-term behaviour of logistic recurrence sequences. You have seen that a relatively simple-looking recurrence relation generates sequences with remarkably rich patterns of behaviour. Other non-linear recurrence relations lead to families of sequences which behave in a similarly complicated way.

More explanation of why some of these patterns arise from such recurrence relations is given in MS221 Chapter B1.

Activity 4.4 An overview of long-term behaviour (Optional)

Open Mathcad file **121B1-02 Overview of logistic recurrence sequences**. Look at the graph plotted in this worksheet. This graph gives an overview of all the types of long-term behaviour that logistic recurrence sequences exhibit for values of the parameter r between 1.5 and 3. Bear in mind that Mathcad could take several seconds to perform the calculations and plot the graph. (Just how long it takes for the graph to be plotted depends on the speed of your computer.)

The graphs plotted in Figures 4.3 and 4.4 were generated with the variable V set to 1500. In this worksheet, V is set to 500 to reduce the time taken to produce data for the graph.

You may like to investigate the types of long-term behaviour displayed for a narrower range of values of r. For example, if you set $R1 = 2.8$ and leave $R2 = 3$, then this region of the graph will be expanded.

If you seek to change *both* of $R1$ and $R2$, then you will find that Mathcad starts to recalculate after the first change has been made. The first two notes in the Comment below explain how this can be avoided.

Can you find a value of r for which the long-term behaviour of the sequence appears to be a 3-cycle?

A solution is given on page 23.

Comment

◇ You can interrupt a Mathcad calculation by pressing `[Esc]`, the escape key, and then clicking 'OK' in the resulting option box. Mathcad will recalculate when you next make a change.

In order to return to automatic mode, follow the same procedure, whereupon the tick mark reappears.

More information about this technique, and details of how to interrupt and resume calculations, are provided in *A Guide to Mathcad.*

- ◇ By default, Mathcad operates in 'automatic calculation mode', but this can be inconvenient where more than one input change is to be made before recalculation is required. In order to switch to 'manual calculation mode' (which disables automatic calculation), select **Automatic Calculation** from the **Math** menu. (When you do this, the tick mark beside **Automatic Calculation** in the menu disappears, and the word 'AUTO' disappears from the status bar in the bottom right corner of the Mathcad window.) Once in manual mode, you can calculate the results as and when you choose, either by selecting **Calculate** from the **Math** menu or by pressing the `[F9]` function key.
- ◇ If your computer is taking a long time to calculate all the terms of the sequences and to plot the graph, you may wish to speed up the process by reducing the amount of computation required. The parameter V determines the number of values of r for which the sequence is calculated, so by decreasing V, the sequence is calculated for fewer values of r, and the computation time is reduced. However, the resulting graph will appear sketchier as a consequence.

Mathcad notes

- ◇ The graph is obtained by plotting $x_{i,n}$ against r_i using the trace type 'points'. The subscripted variable $x_{i,n}$ is obtained in the usual way: either use the 'x_n' button on the 'Matrix' toolbar or type `[` (left square bracket), then separate the subscripts i and n with a comma. Thus you could obtain $x_{i,n}$ on the screen by typing `x[i,n`.
- ◇ A Mathcad graph can display approximately 150 000 individual points. If you attempt to plot a graph with more points than this, then an error may occur. (No graph is drawn and the graph box is highlighted in red – clicking on it reveals the error message 'Can't plot this many points.'.)

Now close file 121B1-02.

Chapter B2, Section 4
Computing with matrices

In this section, you will be working with matrices in Mathcad. The first of the three accompanying files shows how to create matrices and calculate with them in Mathcad. In the second and third files you will study further the long-term behaviour of subpopulations within matrix population models, as introduced in Section 3.

4.1 Matrices in Mathcad

Recall that an $m \times n$ matrix has m rows and n columns. In Activity 4.1 you will learn how to create, define and edit matrices in Mathcad.

See Chapter B2, Section 2.

Activity 4.1 Creating, defining and editing matrices

Open Mathcad file **121B2-01 Matrices in Mathcad**. Read page 2 of the worksheet, follow the instructions, and carry out Task 1.

In the main text, you investigated matrix multiplication, matrix addition and scalar multiplication of matrices. In the next activity, you will see how to use the computer to perform these operations.

See Chapter B2, Section 2.

Activity 4.2 Matrix arithmetic in Mathcad

Read page 3 of the worksheet, and carry out Task 2.

You should still be working with Mathcad file 121B2-01.

If you feel you need, or would like, more practice with multiplying matrices, then read the *optional* final page of the worksheet and perform the matrix multiplications given there.

Solutions are given on page 23.

Mathcad notes

◇ When multiplying two matrices you must enter the multiplication, either by clicking on the multiplication button '×' on the 'Calculator' toolbar or by typing $*$. In contrast, when multiplying a matrix by a scalar (number), Mathcad will insert the multiplication if you omit to enter it. (In both cases, the multiplication appears initially as a small raised dot then, once entry is complete, this dot disappears from view.)

◇ Variables in Mathcad are case-sensitive; that is, upper-case (capital) letters and lower-case (small) letters are regarded as different.

When attempts are made to perform illegal operations on matrices, Mathcad responds with various error messages.

◇ Matrix addition, $\mathbf{A}+\mathbf{B}$, is defined only if the sizes of the two matrices $\mathbf{A}$ and $\mathbf{B}$ are equal, and matrix multiplication, $\mathbf{AB}$, is defined only if the number of columns of the matrix $\mathbf{A}$ is equal to the number of rows of the matrix $\mathbf{B}$. The Mathcad error message for an incorrectly formed matrix sum or product is 'The number of rows and/or columns in these arrays do not match.'.

Mathcad uses the word 'array' to describe a matrix or vector.

◇ Matrix powers, $\mathbf{A}^2$, $\mathbf{A}^3$, ..., are defined only if $\mathbf{A}$ is a square matrix. The Mathcad error message for trying to find powers in other cases is 'This matrix must be square. It should have the same number of rows as columns.'.

◇ If you try to create a 1×1 matrix using the 'Insert Matrix' option box, then the message 'Zero or single element matrix not allowed.' appears in the status bar at the bottom left of the Mathcad window.

Now close file 121B2-01.

You may prefer to omit the remainder of this subsection for the moment, and return to it after studying Chapter B2, Section 5.

You will study matrix arithmetic further in Chapter B2, Section 5. Two important concepts that you will learn about there are the *inverse* of a matrix and the *determinant* of a matrix. Mathcad can be used to find inverses and determinants of matrices, so we include the entry instructions here.

The inverse of a matrix is another matrix of the same size, whereas the determinant is a single number. Both are defined only for square matrices.

The inverse of a (square) matrix is denoted by the matrix raised to the power -1; for example, $\mathbf{A}^{-1}$ denotes the inverse of the square matrix $\mathbf{A}$. To find the inverse of a square matrix in Mathcad, either use the 'X^{-1}' button on the 'Matrix' toolbar or type `^-1`. For example, to display the inverse of the matrix $\mathbf{A}$, you could type `A^-1=`. (Note that the inverse of a square matrix does not always exist; where it does not, Mathcad responds with the error message 'Matrix is singular. Cannot compute its inverse.'.)

To calculate the determinant of a square matrix, either use the '$|X|$' button (available on both the 'Calculator' and 'Matrix' toolbars) or type `[Shift]\` (shift and backslash). For example, to display the determinant of the matrix $\mathbf{A}$, you could type `[Shift]\A=`.

The notations det $\mathbf{A}$, as defined in Chapter B2, Section 5, and $|\mathbf{A}|$ are both commonly used to denote the determinant of $\mathbf{A}$.

A Mathcad display of inverse and determinant, for a particular matrix $\mathbf{A}$, is as follows.

Matrix .... $A := \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix}$

Inverse ...... $A^{-1} = \begin{pmatrix} 0.5 & 0 \\ -1.5 & 1 \end{pmatrix}$

Determinant $|A| = 2$

It is also possible to find the inverse and determinant algebraically, using the buttons '$M^{-1} \rightarrow$' and '$|M| \rightarrow$' on the 'Symbolic' toolbar.

4.2 Matrix population models on the computer

In the main text, you investigated a population model of the UK with two subpopulations: juveniles (aged up to 15 years old) and adults. This model was based upon the sizes of these subpopulations in 1990 and on their birth and death rates. The matrix recurrence relation for this model is

See Chapter B2, Subsection 3.1.

$$\begin{pmatrix} J_{n+1} \\ A_{n+1} \end{pmatrix} = \begin{pmatrix} 0.9326 & 0.0172 \\ 0.0666 & 0.9864 \end{pmatrix} \begin{pmatrix} J_n \\ A_n \end{pmatrix}.$$

Here J_n and A_n represent, respectively, the numbers of juveniles and adults at n years after the starting date in 1990.

You looked at predictions of subpopulation sizes from the model for the years 1990–1997, that is, for $n = 0, 1, \ldots, 7$. This matrix model can be implemented on the computer, which calculates many terms quickly and plots graphs easily, so the long-term behaviour of the model can be observed directly. In the next activity, you will use Mathcad to investigate further the question 'Will there be a constant supply of juveniles entering the potential workforce, or will we have an ever-increasing proportion of adults?'.

Activity 4.3 The two-subpopulation matrix model

Open Mathcad file **121B2-02 Exploring two subpopulations**. Look at page 2 of the worksheet, where the task starts with the two-subpopulation model of the UK population. The parameter N, the number of subsequent years for which the model predicts, is set at 10. Mathcad therefore calculates the subpopulation sizes for the next 10 years, that is, for the years 1991–2000. These subpopulation sizes and the total population size are displayed in tables at the bottom of this page of the worksheet.

The values in the first 8 lines of these tables agree with the values given in Chapter B2, Subsection 3.1, for the years 1990–1997.

Look at page 3 of the worksheet. The first graph shows the sizes of the two subpopulations and the total population. The scale of this graph is quite large, but a slight increase in each of the subpopulations is discernible. The two graphs at the bottom of the page show the two subpopulations separately. Both of these graphs are clearly increasing; however, Mathcad automatically sets the axis limits for both graphs, so the scales are different. The number of juveniles increases by about half a million, whereas the number of adults increases by twice as much. The remaining graph on this page of the worksheet shows that the ratio of successive total populations, T_n/T_{n-1}, is decreasing. A value greater than one for this ratio indicates that the total population is increasing, a value less than one indicates that it is decreasing, and the value one indicates that it is constant year on year. A look at the scale shows that the decrease of the ratio T_n/T_{n-1} here is not significant: it is less than 0.000 02 over the ten years, and the ratio remains greater than one.

This ratio can be expressed as

$$\frac{T_n}{T_{n-1}} = 1 + \frac{T_n - T_{n-1}}{T_{n-1}},$$

that is, one plus the proportionate growth rate (as defined in Chapter B1, Subsection 3.1).

Now look at the graphs on page 4 of the worksheet. The top graph shows the proportions of juveniles and adults relative to the total population; these look fairly constant. The bottom two graphs, which use Mathcad's automatic scaling, look very different; these graphs show the proportion of juveniles increasing and the proportion of adults decreasing. However, a look at the scales reveals that these changes are in fact both less than 0.005 over the ten years.

(a) Return to page 2 of the worksheet, and set the value of N to 50. This changes the value of N throughout the worksheet.

Look at the graphs on page 3. What can you say about the subpopulation sizes and the ratio of successive total populations?

(b) Look at the graphs on page 4. Are the proportions of juveniles and adults eventually fairly constant? Try to answer the question 'Will there be a constant supply of juveniles entering the potential workforce, or will we have an ever-increasing proportion of adults?'.

Confirm your answers by changing N to 100.

Solutions are given on page 23.

Comment

◇ The graphs have been set up to show the total population traces as solid red lines, the adult traces as solid blue lines, and the juvenile traces as solid magenta (purple) lines. If you have any difficulty distinguishing between these colours (especially on the total population graph, where all three are plotted together), or wish to print these graphs on a non-colour printer, then you may like to format the graph to change the line style of some of these traces from 'solid' to 'dot' or to 'dash'.

◇ The long-term behaviour can be seen clearly when N is set to 100; this confirms the trends shown when N is set to 50. However, over such a long period, it is extremely likely that social changes will invalidate the model.

◇ The graphs in this worksheet show how important it is always to look at the scale before drawing any conclusions.

Mathcad notes

◇ The calculation range is $n := 0, 1 \,..\, N - 1$, and the subpopulations calculated at each stage are J_{n+1} and A_{n+1}. Hence the subpopulations $J_1, J_2, \ldots, J_N$ and $A_1, A_2, \ldots, A_N$ are calculated (J_0 and A_0 being specified beforehand). The graph range is $n := 0, 1 \,..\, N$, so the subpopulations $J_0, J_1, \ldots, J_N$ and $A_0, A_1, \ldots, A_N$ are plotted.

◇ You may wonder why the graph range $n := 0, 1 \,..\, N$ is not used to plot the ratio of successive total populations, T_n/T_{n-1}. The denominator, T_{n-1}, is not defined when $n = 0$ (there is no term T_{-1}), so we need to define a separate graph range variable k, taking only the values $1, 2, \ldots, N$, and then plot T_k/T_{k-1}.

◇ Titles have been added to the graphs in this worksheet. This is done by first clicking in the graph to select it, and then choosing **Graph ▶ X-Y Plot...** from the **Format** menu, to bring up the 'Formatting Currently Selected X-Y Plot' option box. The title is entered on the 'Labels' tab in the option box.

In the next activity you will investigate the effect of altering the initial size of the juvenile subpopulation on the long-term behaviour of the two-subpopulation model.

Activity 4.4 Altering the initial juvenile subpopulation size

On page 2 of the worksheet, set $N = 50$, so that the long-term behaviour can be seen.

You should still be working with Mathcad file 121B2-02.

(a) Use the following initial values for the subpopulation sizes:

$$\begin{pmatrix} J_0 \\ A_0 \end{pmatrix} = \begin{pmatrix} 5 \\ 46.49 \end{pmatrix}.$$

Use the tables on page 2 of the worksheet and the graphs on pages 3 and 4 to answer the following questions.

(i) Does the ratio of successive total populations tend to a limit? If so, what is this limit?

(ii) What happens to the numbers of juveniles and adults over the 50-year period of prediction?

(iii) What happens to the proportions of juveniles and adults over the 50-year period of prediction?

(b) Repeat part (a) with the following initial values for the subpopulation sizes:

$$\begin{pmatrix} J_0 \\ A_0 \end{pmatrix} = \begin{pmatrix} 20 \\ 46.49 \end{pmatrix}.$$

(c) Does the long-term behaviour of the model appear to depend on the initial values of the subpopulation sizes?

Solutions are given on page 24.

Comment

◇ In general, the initial values of the subpopulation sizes do not affect the long-term behaviour of the model. This conclusion parallels the results that you discovered in Block A for linear recurrence sequences. You may like to try more substantial changes to the initial subpopulation sizes. The ratio of successive total populations may increase or decrease in the early years, but it always seems to tend to 1.003 (to 3 decimal places), while the proportions of juveniles and adults seem to tend respectively to 0.2 and 0.8 (to 1 decimal place). However, in order to observe this long-term behaviour, the length of time N (years) over which the model is run may need to be increased in some cases to 100 or more.

See Chapter A1, Subsection 6.3.

◇ The accuracy of readings taken from these graphs varies, depending on what scale is chosen by the automatic scaling within Mathcad.

Now close file 121B2-02.

Mathcad is an ideal tool for studying more complicated matrix population models. In the next activity, you will investigate a question posed in the main text: 'Who will support us in our old age?'

Activity 4.5 The three-subpopulation matrix model

Open Mathcad file **121B2-03 Exploring three subpopulations**. The layout of this worksheet is very similar to that of the previous one, with tables giving the values of the subpopulations on page 2, graphs of the subpopulation sizes and the ratio of successive total populations on page 3, and the proportions of the subpopulations on page 4. The obvious difference is that there are three subpopulations included in this model, juveniles (aged up to 15 years old), workers (aged between 15 and 65 years old) and the elderly.

Look at the tables and graphs in the worksheet. At first sight, the outlook predicted by the model is that the number of elderly grows, while the number of workers falls.

Change the number of years for which the model predicts (on page 2 of the worksheet) to $N = 50$.

(a) (i) Does the ratio of successive total populations tend to a limit? If so, what is this limit and what is the consequence for the population as a whole?

(ii) What happens to the proportions of juveniles, workers and elderly over the 50-year period of prediction?

(b) Confirm your observations from part (a) by changing N to 100.

(c) Try to answer the question 'Who will support us in our old age?', by looking at the ratio of the elderly to the workers who will support them.

Solutions are given on page 24.

Comment

You may have been surprised to see that the long-term behaviour of this three-subpopulation model is so different compared to that of the two-subpopulation model, even though the two models are based on the same birth and death rate figures. Recall that the two-subpopulation model predicted long-term population growth at a ratio of 1.003 per year (that is, at a proportionate growth rate of $0.003 = 0.3\%$ per year), while the three-subpopulation model in this activity predicts that eventually the population will decline slowly, at a ratio of about 0.999 per year (a proportionate growth rate of -0.1% per year).

A subtle point that these models do not take into account is that the age distribution *within* a subpopulation may change with time. The three-subpopulation model predicts that the number of elderly will increase as a proportion of all adults (non-juveniles). In the two-subpopulation model, the birth rate proportion is applied to the *whole* adult population, and so with time less account is taken of the fact that an increasing proportion of these adults are over the age of 65, and hence unlikely to reproduce. In the three-subpopulation model, this factor is taken into account and effectively reduces the birth rate of the population as a whole with time, by comparison with the two-subpopulation model.

In practice, models that subdivide the population into much narrower age bands (covering, say, five years each) are used. This helps to avoid such problems.

Mathcad notes

The expressions (normally seen in the middle placeholder of each vertical axis) used to plot the graphs at the bottom of pages 3 and 4 of the worksheet are not shown. Hiding the expressions for the three vertical axes enables us to display the three graphs side by side.

They were hidden by first clicking in each graph to select it, then choosing **Graph ▶ X-Y Plot...** from the **Format** menu and clicking in the check box to 'Hide Arguments' on the 'Traces' tab in the 'Formatting Currently Selected X-Y Plot' option box. However, if you do this, then it is important to add a title to the graph and perhaps to add axis labels too. You can use the settings on the 'Labels' tab in the option box to add these features.

Now close file 121B2-03.

Chapter B3

The main subject matter of this chapter is vectors, in column, component and geometric form, and their application to displacements, velocities and forces. The chapter also includes the Sine Rule and Cosine Rule, together with applications of their use.

There are no specified computer activities to accompany Chapter B3. However, there are possible applications of Mathcad in the context of the chapter, and you are encouraged to put Mathcad to use where it seems appropriate and convenient to do so. Here are some specific suggestions on where Mathcad might be applied.

◇ If a vector is given in geometric form, in terms of a magnitude and direction, then Mathcad can be used to find its components. This is illustrated below for the case where the vector has magnitude $m = 4$ and direction $\theta = 120°$.

Geometric form	Magnitude	$m := 4$	Direction	$\theta := 120\text{deg}$
Component form	i-component	$m\cos(\theta) = -2$		
	j-component	$m\sin(\theta) = 3.464$ (to 3 d.p.)		

Note that Mathcad interprets input to any trigonometric function as being in radians. To work with angles in degrees, it is necessary to multiply by the built-in Mathcad constant $deg = \pi/180$, as shown above. (This converts from degrees to radians.)

◇ If a vector is given in component form, then Mathcad can be used to find its magnitude and to assist in finding its direction. This is illustrated below for the vector whose component form is $\sqrt{3}\mathbf{i} + \mathbf{j}$, with corresponding components $a1 = \sqrt{3}$ and $a2 = 1$.

Component form	i-component	$a1 := \sqrt{3}$	j-component	$a2 := 1$
Geometric form	Magnitude	$\sqrt{a1^2 + a2^2} = 2$		
	Direction (obtained from)	$\phi := \dfrac{\operatorname{atan}\left(\left\|\dfrac{a2}{a1}\right\|\right)}{deg}$		$\phi = 30$

Mathcad gives an error if $a1 = 0$ is used here.

The arctan function was introduced in Chapter A3, Subsection 4.2 (along with arcsin and arccos, referred to overleaf).

The output for the angle ϕ is found using the arctan function, denoted in Mathcad by atan. This output will be in radians, and the corresponding value in degrees is obtained on dividing by *deg*, as shown. There remains the step of finding the direction θ (in the range $-180° < \theta \leq 180°$) from ϕ (between 0 and 90°). This is achieved as explained in Figure 2.5 and surrounding text in Chapter B3, and is not a task which can be performed immediately by Mathcad.

◇ Use of the Sine Rule or Cosine Rule to find an angle θ of a triangle leads to an equation of the form $\sin\theta = \ldots$ or $\cos\theta = \ldots$, respectively. Such an equation can be solved for θ using the function arcsin (asin in Mathcad) or arccos (acos in Mathcad). However, you need to bear in mind that an equation $\sin\theta = \ldots$ or $\cos\theta = \ldots$ usually has more than one solution within the range $-180° < \theta \leq 180°$. The function arccos gives the solution for which $0 \leq \theta \leq 180°$, and so always provides the correct answer for an angle within a triangle. On the other hand, arcsin gives the solution for which $-90° \leq \theta \leq 90°$ or, for an angle within a triangle, $0 < \theta \leq 90°$. You then need reasoning independent of Mathcad to decide whether it is the given output or 180° minus that output which is the angle required. Further details on this point are given in Subsection 3.1 of the chapter.

As with atan, the Mathcad functions asin and acos give output in radians, which can be converted to degrees on division by *deg*.

Recall that

$$\sin(180° - \theta) = \sin\theta.$$

Mathcad notes

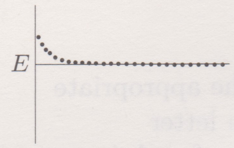
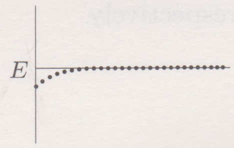
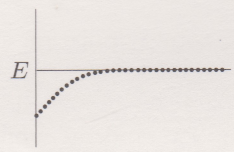
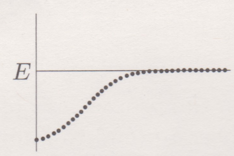
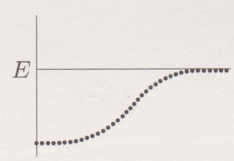
To enter Greek letters in Mathcad, you can either click on the appropriate button on the 'Greek' toolbar, or type the equivalent Roman letter followed by `[Ctrl]g`. For example, to obtain the Greek letters θ and ϕ, you can click on their buttons or type `q[Ctrl]g` and `f[Ctrl]g`, respectively.

Solutions to Activities

Chapter B1

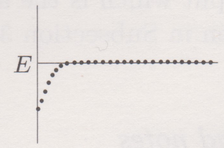
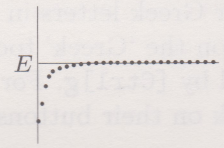
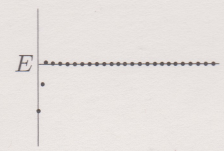
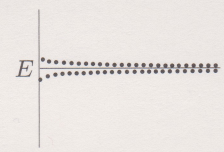
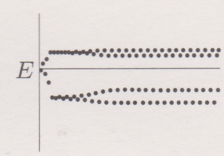
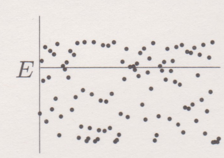
Solution 4.1

Here $r = 0.2$ and $E = 13\,300$. The descriptions, in words and sketches, are shown below. The sequences all tend to E in the long run.

P_0		
20 000	decreasing tends to E	
10 000	increasing tends to E	
5 000	increasing tends to E	
1 000	increasing tends to E slight S-shape	
100	increasing tends to E S-shape	

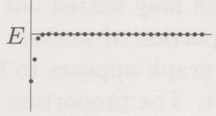
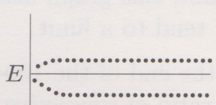
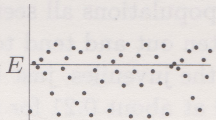
Solution 4.2

Here $P_0 = 5000$ and $E = 13\,300$. The descriptions are shown below.

r		
0.5	increasing tends to E	
1	increasing tends to E rapidly	
1.5	tends to E terms above and below E	
2	tends to E slowly terms above and below E	
2.5	has no limit takes one of four values	
3	random values between two bounds	

Solution 4.3

Again $P_0 = 5000$ and $E = 13\,300$. The descriptions are shown below.

r		
1.25	tends to E rapidly terms above and below E	
2.25	has no limit takes one of two values	
2.75	random values between two bounds	

Solution 4.4

The long-term behaviour of the sequence appears to be a 3-cycle for values of r between about 2.83 and 2.84.

Chapter B2

Solution 4.2

The answers to Task 2 are as follows:

$$\mathbf{AB} = \begin{pmatrix} 8 & 12 \\ 14 & 17 \end{pmatrix}, \qquad \mathbf{BA} = \begin{pmatrix} 26 & 6 \\ 1 & -1 \end{pmatrix},$$

$$\mathbf{A}^2 = \begin{pmatrix} 4 & 0 \\ 9 & 1 \end{pmatrix}, \qquad \mathbf{A}^3 = \begin{pmatrix} 8 & 0 \\ 21 & 1 \end{pmatrix},$$

$$\mathbf{A}+\mathbf{B} = \begin{pmatrix} 6 & 6 \\ 5 & 0 \end{pmatrix}, \qquad \mathbf{A}-\mathbf{B} = \begin{pmatrix} -2 & -6 \\ 1 & 2 \end{pmatrix},$$

$$1.5\mathbf{A} = \begin{pmatrix} 3 & 0 \\ 4.5 & 1.5 \end{pmatrix}.$$

The following are the answers to the *optional* question. Nine products can be formed:

$$\mathbf{CC} = \begin{pmatrix} 200 & -16 \\ -64 & 8 \end{pmatrix}, \qquad \mathbf{Cx} = \begin{pmatrix} -121 \\ 26 \end{pmatrix},$$

$$\mathbf{CD} = \begin{pmatrix} -126 & 20 & 96 & -149 \\ 36 & -16 & -24 & 34 \end{pmatrix},$$

$$\mathbf{EE} = \begin{pmatrix} 68 & 67 & -117 \\ -76 & -42 & -35 \\ -107 & -11 & 158 \end{pmatrix},$$

$$\mathbf{EF} = \begin{pmatrix} 7 & 64 \\ -26 & -44 \\ -4 & 25 \end{pmatrix}, \qquad \mathbf{Ev} = \begin{pmatrix} 147 \\ -80 \\ -35 \end{pmatrix},$$

$$\mathbf{FC} = \begin{pmatrix} -8 & -8 \\ 48 & -12 \\ 2 & -7 \end{pmatrix}, \qquad \mathbf{Fx} = \begin{pmatrix} 43 \\ 7 \\ 29 \end{pmatrix},$$

$$\mathbf{FD} = \begin{pmatrix} 18 & 28 & -24 & 47 \\ 18 & 32 & 4 & 3 \\ 9 & 23 & -15 & 31 \end{pmatrix}$$

Solution 4.3

(a) The total population and both of the subpopulations increase over the 50-year period; the number of juveniles increases by about 2 million, and the number of adults increases by about 7 million. The ratio of successive total populations decreases, but its graph flattens out. It appears to tend to a limit just above 1.002 73.

(b) Yes, the proportions of juveniles and adults both appear to tend to limits, 0.197 and 0.803, respectively (to 3 decimal places), so they are eventually fairly constant. Therefore the answer to the question is that there will be a constant proportion of juveniles joining the potential workforce, in fact, $0.197/15 \simeq 0.013$ or 1.3% of the population each year.

Changing N from 50 to 100 confirms these results.

Solution 4.4

(a) (i) Yes, the ratio of successive total populations decreases and appears to tend to a limit between 1.0027 and 1.0028.

(ii) The number of juveniles increases over the 50-year period of prediction, faster at first. The number of adults decreases at first before gradually increasing. The model predicts an increase of about 7 million juveniles and an increase of about 1 million adults by the end of the 50-year period of prediction.

(iii) The proportion of juveniles increases and appears to tend to a limit just below 0.2. The proportion of adults decreases and appears to tend to a limit just above 0.8.

(b) (i) Yes, the ratio of successive total populations increases and appears to tend to a limit just above 1.0027.

(ii) The number of juveniles decreases at first before gradually increasing. The number of adults increases, faster at first. The model predicts a decrease of about 5 million juveniles and an increase of about 15 million adults by the end of the 50-year period of prediction.

(iii) The proportion of juveniles decreases and appears to tend to a limit just below 0.2. The proportion of adults increases and appears to tend to a limit just above 0.8.

(c) For both sets of initial values for the subpopulation sizes, the ratio of successive total populations appears to tend to a limit between 1.0027 and 1.0028, and the proportions of juveniles and adults seem to tend to limits just below 0.2 and just above 0.8, respectively. The actual numbers of juveniles and adults are different for the two cases, but the long-term behaviour of the model does not appear to depend on the initial values of the subpopulation sizes.

Solution 4.5

(a) (i) The ratio of successive total populations appears to tend to a limit just below 0.999. This ratio is less than one, so the prediction is for a slight year-on-year decrease in the total population.

(ii) The proportion of juveniles increases at first, then decreases slightly; it looks as if the graph may flatten out and tend to a limit. The proportion of workers decreases, faster at first; the graph appears to flatten out and tend to a limit. The proportion of elderly increases, faster at first; this graph also appears to flatten out and tend to a limit.

(b) By the end of the 100-year period of prediction, the ratio of successive total populations still appears to tend to a limit, between 0.998 and 0.999. The proportions of the subpopulations all seem to have graphs that flatten out and tend to limits, at about 0.1925 for the juveniles, just below 0.6 for the workers, and at about 0.21 for the elderly.

(c) The model predicts that, at the end of the 50-year period, the ratio of the elderly to the workers supporting them is about 1:3. It appears therefore that a stable and sustainable situation may be reached, in terms of support of the elderly by the workers.

(For example, if a state pension is to be one half of the average working wage, then the fraction of each worker's wage needed to support pensions for the elderly is predicted to be $\frac{1}{3}$ of $\frac{1}{2}$, that is, $\frac{1}{6}$, or about 17%.)

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